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On the existence of horizons in spacetimes with vanishing curvature invariants

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Abstract: A direct very simple proof that there can be no closed trapped surfaces (ergo no black hole regions) in spacetimes with all curvature scalar invariants vanishing is given. Explicit examples of the recently introduced "dynamical horizons" which nevertheless do not enclose any trapped region are presented too.

Keywords: Black Holes in String Theory, Space-Time Symmetries, Classical Theories of Gravity, Differential and Algebraic Geometry.

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There is a renewed interest on (generalized) pp-wave spacetimes.[1] This is due to the fact that all scalar invariants constructable from the Riemann tensor and its derivatives vanish in pp-wave spacetimes, which implies that they are *exact* solutions of the full non-linear classical string theory [4]. Moreover, it is known since long ago that every spacetime can be "approximated", nearby a null geodesic, to a plane wave (a particular case of pp-waves, see e.g. [1, 5, 6]), which is called the Penrose limit of the spacetime on that geodesic [7]. This has relevant consequences and applications in the context of the AdS/CFT correspondence, see e.g. [8, 9] and references therein. The above led to the analysis of the pp-wave conformal boundary, linked via AdS/CFT to the associated conformal theory, with the remarkable result that pp-wave spacetimes seem to have a single null line as conformal boundary [10, 3].

More generally, spacetimes with all scalar invariants vanishing (VSI from now on) share the above mentioned property of being exact solutions of classical string theory [4, 11]. Thus, they are claimed to give some insight into an acceptable theory of quantum gravity. The question has been raised of whether these exact solutions of string theory can describe, or contain, black-hole regions, something which would be expected to provide deeper or clearer hints on the path to the sought quantum theory of gravitational fields. As a matter of fact, using techniques of geodesic connectivity, one can demonstrate that (some) pp-waves can have no "event horizon" [12] — in the sense that every point in the spacetime can be joined to infinity by means of a causal future-directed curve —, see also [13]. However, the question remains open for general VSI spacetimes, as has been recently pointed out in [14].

In this short note, I address the issue and give a direct very simple proof of the complete absence of closed trapped surfaces (and more generally of closed trapped submanifolds of any co-dimension) in VSI spacetimes. Interestingly, the proof relies on recently developed (i) concepts generalizing Killing vectors — Kerr-Schild symmetries [15] — and (ii) arguments on the interplay between symmetries and trapped surfaces [16]; this is indication of potential applications of both (i) and (ii) to general theories based on lorentzian geometry. I intend to call attention to these works by means of the particular application treated herein. The proof will be essentially geometric and no field equations or any other conditions are assumed. As a by-product, closed *marginally* trapped surfaces will come out to be ubiquitous in VSI spacetimes, and thereby explicit examples of *dynamical horizons* in the sense of [17] will be exhibited, showing that this kind of horizons do not enclose trapped regions or black holes in general.

Consider the VSI spacetime (signature $-, +, \dots, +$), in arbitrary dimension D, given in local coordinates $\{u, v, x^i\}$ $(i, j, \dots = 1, \dots, D-2)$ by the line-element [11]

$$ds^2 = -2du(dv + Hdu + W_i dx^i) + g_{ij} dx^i dx^j \tag{1}$$

where $H = H(u, v, x^i)$ is arbitrary, the functions $g_{ij} = g_{ji} = g_{ij}(u, x^k)$ are independent of

[1] See e.g. [1] and references therein for the standard definition and properties of pp-waves in General Relativity, and [2, 3] for generalizations to arbitrary dimensions.

v, and W_i are linear on v, that is

$$W_i = vF_i' + Z_i(u, x^k)$$

the v-coefficient F_i' depending only on the corresponding x^i (primes stand for derivatives with respect to the argument).[2] There is a preferred null vector field $\vec{\ell} = \partial_v$ in these spacetimes, characterized by being a gradient (hence geodesic), shear-free and expansion-free. Its covariant form is $\boldsymbol{\ell} = -du$. Therefore, (1) belongs to the so-called Kundt class (generalized to D dimensions) [18, 1]. The pp-waves are characterized by admitting the existence of a covariantly constant null vector field $\vec{\ell}$, whence they are included in (1) when $F_i' = 0$ and $H_{,v} = 0$ — a comma indicates partial derivative.

In general, though, the null vector field $\vec{\xi} \equiv \exp\left(-\sum_i F_i\right)\vec{\ell}$ still satisfies a quite interesting relation, namely

$$\pounds_{\vec{\xi}}\, g = -2F^{-1} H_{,v}\, \boldsymbol{\ell} \otimes \boldsymbol{\ell}$$

where $\pounds_{\vec{\xi}}$ stands for the Lie derivative with respect to $\vec{\xi}$ and $F > 0$ is a shorthand for $\exp\left(\sum_i F_i\right)$. This is the differential condition defining the so-called *Kerr-Schild vector fields* [15], which are the generators of one-parameter groups of Kerr-Schild transformations. A study of these infinitesimal symmetries, their properties, and some applications can be found in [15]. Even more importantly, the crucial point in what follows is that, actually, there is an *infinite number* of Kerr-Schild vector fields, all proportional to $\vec{\ell}$, depending on an arbitrary function of u. This follows straightforwardly from Lemma 3.2 and Theorem 2 in [15], or alternatively by a direct calculation:

$$\pounds_{\vec{\xi}_f}\, g = -2F^{-1}\left[f(u)H_{,v} + f'(u)\right] \boldsymbol{\ell} \otimes \boldsymbol{\ell} \tag{2}$$

for all vector fields

$$\vec{\xi}_f \equiv f(u)\, F^{-1}\, \vec{\ell} \tag{3}$$

$f(u)$ being an *arbitrary* function — thus, the Lie algebra of Kerr-Schild vector fields with respect to $\vec{\ell}$ is infinite-dimensional. Let us remark that the above statement holds for pp-wave spacetimes too (including plane waves and even flat spacetime): of course, there are not Killing vectors depending on arbitrary functions, but there certainly are Kerr-Schild vector fields with that property, and they happen to include, for a choice of the arbitrary function $f(u)$ (up to a constant of proportionality), a Killing vector.

The arguments of [16] can now be applied. Let S be any spacelike submanifold of any dimension d, and let $\{\vec{e}_A\}$ $(A, B, \ldots = 1, \ldots, d)$ denote a set of d linearly independent tangent vector fields to S on V. Let $\gamma_{AB} = g|_S(\vec{e}_A, \vec{e}_B)$ be the first fundamental form inherited by S and $\overline{\nabla}$ its canonical connection. In [16] we proved that, for *any* vector field $\vec{\xi}$,

$$\frac{1}{2}\gamma^{AB}\pounds_{\vec{\xi}}\, g|_S(\vec{e}_A, \vec{e}_B) = \overline{\nabla}_C \overline{\xi}^C + (\xi_\mu \mathcal{H}^\mu)|_S \tag{4}$$

[2] By using the remaining coordinate freedom [11, 1], one can rewrite the above such that $W_1 = vF_1'(x^1) + Z_1(u, x^k)$ with $F_1'(x^1) = dF_1(x^1)/dx^1$, and $W_j = Z_j(u, x^k)$ for all $j \neq 1$. However, this is not necessary here. Analogously, one could allow for an arbitrary function in front of dv in (1).

where for all $\vec{v}$, $\overline{v}_C = v_\mu|_S\, e^\mu_C$, and $\vec{\mathcal{H}}$ denotes the mean curvature vector of S (e.g. [19, 20, 21, 16]). Future trapped submanifolds are characterized by having $\vec{\mathcal{H}}$ pointing to the future all over S, and similarly for past trapped. The trapping is proper, near, or marginal according to whether $\vec{\mathcal{H}}$ is timelike, non-spacelike, or null and non-vanishing all over S [16, 5, 19]. Therefore, if S is trapped and $\vec{\xi}$ is future (or past) pointing all over S, the second term on the righthand side of (4) cannot change sign.

Suppose, then, that S is (marginally, nearly) future trapped and closed (i.e. compact without boundary; this is the case of interest for black-hole regions). Replacing the Kerr-Schild symmetries (3) for $\vec{\xi}$ in (4) — with $f(u)$ positive, say, so that $\vec{\xi}_f$ are future pointing — integrating the resulting relation on S, using (2) and Gauss' theorem, one arrives at

$$-\int_S \left(\gamma^{AB}\overline{\ell}_A\overline{\ell}_B\right) F^{-1}\left[f(u)H_{,v} + f'(u)\right]\boldsymbol{\eta}_S = \int_S F^{-1} f(u)(\ell_\mu \mathcal{H}^\mu)\,\boldsymbol{\eta}_S \leq 0 \qquad (5)$$

where $\boldsymbol{\eta}_S$ is the canonical d-volume element on S. Recalling that $f(u)$ is an arbitrary function, this inequality clearly leads to a contradiction in general. To prove it rigourously, notice that $\gamma^{AB}\overline{\ell}_A\overline{\ell}_B \geq 0$, and this vanishes if and only if $\overline{\ell}_A = \ell_\mu|_S e^\mu_A = 0$. If $\overline{\ell}_A$ were non-zero at some points of S, one could always choose $f(u)$ such that $(f(u)H_{,v} + f'(u))\,|_S \leq 0$ which would give the "wrong" sign for the left integral in (5). Indeed, as S is compact, $H_{,v}$ will reach its maximum M on S, hence $H_{,v}|_S \leq M$, so that it would be enough to choose a positive f with $f'/f \leq -M$ (for instance, $f(u) = e^{-Mu}$ would do). The only possibility is therefore that $\overline{\ell}_A = 0$. Then the right integral of (5) must vanish as well, whence $\ell_\mu\mathcal{H}^\mu = 0$. As $\vec{\mathcal{H}}$ and $\vec{\ell}$ are causal this implies that in fact they must be proportional to each other. In summary

> *The VSI spacetimes (1) do not admit closed trapped or nearly trapped submanifolds of any dimension. And any closed marginally trapped submanifold must have $\vec{\mathcal{H}} \propto \vec{\ell}$ and be contained in one of the null hypersurfaces $u =$const. orthogonal to the null vector field $\vec{\ell}$.*

Remark: observe that this result is valid for *completely* general pp-waves as well, as was proved already in [16].

The meaning of the general result, when applied to submanifolds of co-dimension 2, is that there cannot be event horizons for asymptotically flat cases, if any. More generally, the absence of closed trapped surfaces implies that there are no "trapping horizons" in the sense of [22]. These are horizons defined locally, without any reference to infinity, describing the boundary of black (or white) holes, see [22]. Thus, there cannot be black hole regions in the VSI spacetimes.

The above leads also to the absence of *apparent horizons* — see e.g. [5, 23] for the asymptotically flat case and [19] for the general case — which is defined roughly as boundary of the set of closed trapped surfaces in the spacetime, and itself defines (under certain assumptions of continuity) closed marginally trapped surfaces [19, 23]. Let us remark, however, that the recently proposed definition of "dynamical horizons" [17], trying to improve that of trapping horizons, does not capture the absence of black holes in the VSI spacetimes. As a matter of fact, one can find many examples of dynamical horizons in

VSI spacetimes. To see this, observe that *all* surfaces $S_{u,v}$ of co-dimension 2 given by constant values of the coordinates u and v are marginally trapped, as follows from a trivial calculation using for instance the formulas presented in [21] applied to (1):

$$\vec{\mathcal{H}} = \left(U_{,u} - \mathrm{div}\vec{W}\right)\Big|_{S_{u,v}} \vec{\ell}.$$

Here $U = \log\sqrt{\det g_{ij}}$, $\boldsymbol{W} = W_j dx^j$, and div is the divergence on each $S_{u,v}$. Hence, the mean curvature vector of these surfaces is always null and the expansion θ_ℓ corresponding to the null normal $\vec{\ell}$ vanishes. Take then, for example, the hypersurfaces $\Sigma: \ v = h(u)$. These hypersurfaces are spacelike if $(2H + \hat{g}^{ij}W_iW_j)|_\Sigma < 2\dot{h}$, where $\hat{g}^{ij}$ is the inverse matrix of g_{ij}, and they are foliated by the marginally trapped surfaces $S_{u,v}$. Choosing these to be closed, which is obviously possible, all conditions in the definition of dynamical horizons hold for Σ by choosing $(U_{,u} - \mathrm{div}\vec{W})|_\Sigma$ to be strictly positive (this is minus the "inwards" expansion). Note that this is valid even for the simpler pp-waves.[3]

There remains the question of validity of the local coordinates, or whether or not the line-elements (1) are extensible,[4] see e.g. [5, 24]. In the case of usual pp-waves the coordinates are globally defined, as is known, and the spacetime is geodesically complete. See [13] for a recent discussion of when this can be generalized to some subcases of (1) with $W_i = 0 = g_{ij,v}$. In the general case (1), problems may arise depending on the explicit form of the functions H, W_i and g_{ij}. The vanishing of $\det g_{ij}$ at some points will usually indicate either a curvature singularity or a problem with the completeness of the spacelike submanifolds spanned by the $\{x^i\}$, and occasionally extensibility of causal geodesics. Similarly, if F_i' or H reach unbounded values then a curvature singularity appears generically. There may be, though, particular cases in which the spacetime is extensible. However, if the spacetime were extensible *keeping the VSI form* (1), then a similar argument would apply to the added region. While if the VSI property is lost through the extension, then this is no longer an exact solution of the classical string theory and the question loses its main interest.

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[3]Nevertheless, Σ are not trapping horizons, as the derivative of the vanishing expansion along the inward null direction vanishes. So, trapping horizons seem to be better adapted to rule out cases like the VSI spacetimes.

[4]Note, for instance, that the exterior $r > 2m$ region of Schwarzschild solution is static, so that similar arguments apply [16] and the absence of closed trapped surfaces is obvious. The trapped surfaces of the Schwarzschild black hole appear in the added region after the spacetime has been extended.

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